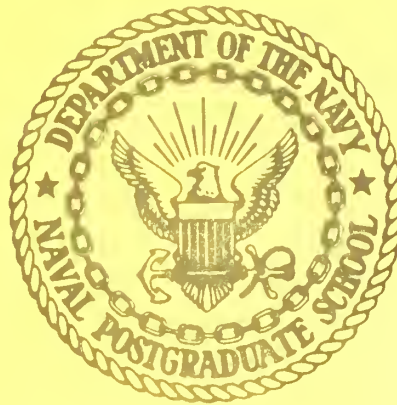NPS55-86-019

NAVAL POSTGRADUATE SCHOOL
Monterey, California

A BIVARIATE FIRST ORDER AUTOREGRESSIVE
TIME SERIES MODEL IN EXPONENTIAL
VARIABLES (BEAR(1))

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OCTOBER 1986

Prepared for:
Naval Postgraduate School
Monterey, CA 93943-5000

NAVAL POSTGRADUATE SCHOOL
Monterey, California

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REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS			
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.			
2b DECLASSIFICATION / DOWNGRADING SCHEDULE				
4 PERFORMING ORGANIZATION REPORT NUMBER(S) NPS55-86-019	5 MONITORING ORGANIZATION REPORT NUMBER(S)			
6a NAME OF PERFORMING ORGANIZATION Naval Postgraduate School	6b OFFICE SYMBOL (If applicable) Code 55	7a. NAME OF MONITORING ORGANIZATION		
6c ADDRESS (City, State, and ZIP Code) Monterey, CA 93943-5000	7b ADDRESS (City, State, and ZIP Code)			
8 NAME OF FUNDING/SPONSORING ORGANIZATION	8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER		

ADDRESS (City, State, and ZIP Code)	10 SOURCE OF FUNDING NUMBERS			
	PROGRAM ELEMENT NO	PROJECT NO	TASK NO.	WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)
BIVARIATE FIRST ORDER AUTOREGRESSIVE TIME SERIES MODEL IN EXPONENTIAL VARIABLES (BEAR(1)) (UNCLASSIFIED)

12 PERSONAL AUTHOR(S)
Dewald, L. S., Lewis, P. A. W. and McKenzie, E.

13a. TYPE OF REPORT Technical	13b TIME COVERED FROM _____ TO _____	14 DATE OF REPORT (Year, Month, Day) October 1986	15. PAGE COUNT 32

16 SUPPLEMENTARY NOTATION

17	COSATI CODES		18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	time series; bivariate exponential distribution; Autoregressive models; NEAR(2); ARMA(2.1) models; Gaussian AR(1) bivariate time series model

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

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20 DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a NAME OF RESPONSIBLE INDIVIDUAL Peter A. W. Lewis	22b TELEPHONE (Include Area Code) (408) 646-2283	22c OFFICE SYMBOL Code 55Lw

DD FORM 1473, 84 MAR

83 APR edition may be used until exhausted
All other editions are obsolete

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IN EXPONENTIAL VARIABLES (BEAR(1))

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ABSTRACT

A simple time series model for bivariate exponential variables having first-order autoregressive structure is presented. The linear random coefficient difference equation model is an adaptation of the New Exponential Autoregressive model (NEAR (2)). The process is Markovian in the bivariate sense and has correlation structure analogous to that of the Gaussian AR(1) bivariate time series model. The model exhibits a full range of positive correlations and cross-correlations. With some modification in either the innovation or the random coefficients, the model admits some negative values for the cross-correlations. The marginal processes are shown to have correlation structure of ARMA(2,1) models.

Key Words: Time Series; Bivariate Exponential Distribution; Autoregressive Models; NEAR(2); ARMA(2,1) Models; Gaussian AR(1) Bivariate Time Series Model.

1. INTRODUCTION

The homogeneous Poisson process is a basic model for point processes (series of events) and can be characterized as a process in which the intervals between events are independent and identically exponentially distributed random variables. One generalization of the homogeneous Poisson process has been to relax the condition on independence of intervals. The earliest attempts to do this go back to Wold (1948) and Cox (1955). See Cox and Lewis (1966, Ch 7) for details. Later, this

need for point processes with correlated but marginally exponential distributions inspired the creation of a class of ARMA time series models for the univariate exponential processes analogous to that for the Gaussian processes (Gaver and Lewis, 1980; Lawrance and Lewis, 1981, 1985). Beyond their utility in point processes, these models are utilized in studying queues (Jacobs, 1978; 1980), inventory and water resources problems, and other situations where non-negative random variables are appropriate for inputs. Using antithetic variables as innovation or error structure, these models also allow for negative dependence (see Lewis, 1986, for a survey).

We demonstrate that one of the more recent exponential models with ARMA correlation strucutre, NEAR(2), by Lawrance and Lewis (1985), lends itself naturally to a first-order bivariate autoregressive process which is Markovian in the bivariate sense, and has a structure analogous to that of the Gaussian AR(1) bivariate time series model as given in Tiao and Box (1981), and elsewhere. Early attempts to construct bivariate exponential time series and bivariate Poisson processes are discussed in Cox and Lewis (1971). The present model is broader and simpler than any previously obtained, including the MEAR(1) model by Raftery (1982), which turns out to be a special case of the present model for the bivariate case, and the models of Jacobs (1978, 1980), Lewis and Shedler (1977), and Gaver and Lewis (1980). All of these models were defined using the NEAR(1) structure, and this does not allow for the breadth obtained by using the NEAR(2) structure.

With the general model it is, for example, possible to construct simple bivariate, serially correlated models for the successive up and

down times which occur in reliability studies, or for correlated and cross-correlated service and interarrival sequences in queueing systems.

2. THE UNIVARIATE NEAR(2) MODEL

By the NEAR(2) theorem (Lawrance and Lewis, 1985), we know that two (possibly dependent) random variables with marginally Exponential(λ) distributions can be combined with three scaled (possibly dependent) exponentials which are independent of the first pair, to give another random variable that is marginally exponentially distributed with parameter λ. We assume in the remainder of this presentation that $\lambda = 1$. The essence of the NEAR(2) theorem is repeated without proof.

THEOREM (Lawrance and Lewis)

If $\{E_n\}$ is an i.i.d. sequence of exponential random variables and the parameters α_1, β_1, α_2, β_2 are such that $0 < \alpha_i < 1$ for i=1,2; $0 < \alpha_1 + \alpha_2 < 1$ and $0 < \beta_1$, $\beta_2 < 1$ and if $X_0 = E_0$, then

$$
X_n = \begin{bmatrix} \beta_1 X_{n-1} & \text{w.p.} & \alpha_1 \\ \beta_2 X_{n-2} & \text{w.p.} & \alpha_2 \\ 0 & \text{w.p.} & 1-\alpha_1-\alpha_2 \end{bmatrix} + \begin{bmatrix} E_n & \text{w.p.} & 1-p_2-p_3 \\ b_2 E_n & \text{w.p.} & p_2 \\ b_3 E_n & \text{w.p.} & p_3, \end{bmatrix} \quad (2.1)
$$

defines a stationary sequence $\{X_n\}$ which is marginally exponentially distributed with $\lambda=1$. The quantities b_2, b_3, p_2, p_3 have values in the interval $(0,1)$ and are defined in Lawrance and Lewis (1985); the mixture of the exponential random variables on the right in (2.1) is called the innovation random variable.

A special case of the NEAR(2) model which is not covered by the theorem, but obtained directly from the usual analysis with transforms, is the case when $\beta_1 = \beta_2 = 1$. In this case, still assuming $\alpha_1 + \alpha_2 < 1$, we have

$$
X_n = \begin{bmatrix} X_{n-1} & \text{w.p.} & \alpha_1 \\[2mm] X_{n-2} & \text{w.p.} & \alpha_2 \\[2mm] 0 & \text{w.p.} & 1-\alpha_1-\alpha_2 \end{bmatrix} + (1-\alpha_1-\alpha_2)E_n. \qquad (2.2)
$$

This is the TEAR(2) model. It has two advantages for our purposes; one that it only uses two parameters instead of four in the complete NEAR(2) model; and the other that it has a simpler innovation structure. The model exhibits, however, a "runs-up" behavior in the sample paths. Nevertheless, we will use the TEAR(2) model as the base for construction of the bivariate exponential model, even though other three- and two-parameter subclasses of the NEAR(2) process may be phenomenologically more suitable. For example, in hydrology the time series which occur generally exhibit a "runs-down" behavior. However, the ideas in developing the bivariate process are the same.

3. BIVARIATE, MARKOVIAN TIME SERIES FOR EXPONENTIAL RANDOM VARIABLES

Let $\{X_n, Y_n\}$ be a bivariate sequence of random variables, such that $\{X_n\}$ and $\{Y_n\}$ are each stationary sequences of marginally exponentially distributed random variables with $\lambda=1$. Then we define our first bivariate exponential model, using (2.2), as

$$
X_n = K_{11}^{(n)}X_{n-1} + K_{12}^{(n)}Y_{n-1} + (1-\alpha_{11}-\alpha_{12})E_n \qquad (3.1)
$$

and

$$Y_n = K_{21}^{(n)} X_{n-1} + K_{22}^{(n)} Y_{n-1} + (1-\alpha_{22}-\alpha_{21})E_n; \qquad (3.2)$$

in this equation

$$\{K_{11}^{(n)}, K_{12}^{(n)}\} = \begin{bmatrix} (0,1) & \text{w.p.} & \alpha_{12} \\ (1,0) & \text{w.p.} & \alpha_{11} \\ (0,0) & \text{w.p.} & 1-\alpha_{11}-\alpha_{12} \end{bmatrix}$$

and are serially independent;

$$\{K_{21}^{(n)}, K_{22}^{(n)}\} = \begin{bmatrix} (0,1) & \text{w.p.} & \alpha_{22} \\ (1,0) & \text{w.p.} & \alpha_{21} \\ (0,0) & \text{w.p.} & 1-\alpha_{21}-\alpha_{22} \end{bmatrix} ,$$

are also serially independent and independent of $\{K_{11}^{(n)}, K_{12}^{(n)}\}$ for all
n. Likewise $\{E_n\}$ and $\{E_n'\}$ are independent innovation sequences of
i.i.d. Exponential($\lambda=1$) random variables. Finally, we insist that
$\alpha_{11} + \alpha_{12} < 1$ and $\alpha_{22} + \alpha_{21} < 1$. If $\alpha_{11} + \alpha_{12} = 1$ and
$\alpha_{22} + \alpha_{21} = 1$, then the process is not ergodic. In this case, $\{X_n, Y_n\}$
is always one of the pairs (E_1, E_1'), (E_1, E_1), (E_1', E_1') (E_1', E_1) for all
n. Note that X_{n-1} and Y_{n-1} in, e.g., Equation (3.1) are not
independent; this is the reason for the use of the NEAR(2) construction.

We observe that $\{X_n\}$, $\{Y_n\}$ are each TEAR(2) constructions in that
Y_{n-1} in (3.1) replaced X_{n-1} in (2.1), and similarly X_{n-1} in (3.2)
replaced Y_{n-2}. Also, (X_n, Y_n) is, given the value of (X_{n-1}, Y_{n-1}),
completely independent of previous values (X_k, Y_k) for $k \leq n-2$ for

all n. Again $\mathcal{E}(X_n | X_{n-1} = x, Y_{n-1} = y)$ is a linear function of (x,y)
and likewise for Y_n. Thus, the model is Markovian in the bivariate
sense, both structurally and in the sense of expectations.

Note, however, that $\{X_n\}$ and $\{Y_n\}$ are <u>not</u> marginally TEAR(1)
processes, and, in fact, are not marginally Markovian. In particular,
their correlations are not necessarily geometrically decaying, as will
be demonstrated later. If $\alpha_{11} = \alpha_{22} = 0$, we obtain the bivariate
exponential process described by Gaver and Lewis (1980). The process
$\{Y_n\}$ is then called the <u>dual</u> of the process $\{X_n\}$.

Clearly, other special combinations of the α_{ij} give other models
with 1, 2, or 3 parameters. Of course, there could be from 5 to 8 total
parameters by using the more general NEAR(2) construction. Thus, from
(2.1), (3.1), and (3.2), replace $K_{ij}^{(n)}$ by $\beta_{ij} K_{ij}^{(n)}$, $0 < \beta_{ij} < 1$. One
requires then, however, the more complicated innovation structure given
in (2.1) for the innovation in (3.1) and (3.2).

4. CORRELATION STRUCTURE OF THE MODEL

The marginal processes defined above are exponentially distributed
with mean $1/\lambda$ and variance $1/\lambda^2$. Thus, if we set the scale parameter
λ equal to 1, correlations and covariance are equal. Since we assume
$\{Y_n\}$, $\{X_n\}$ are jointly stationary, we have

$$\gamma_{XX}(\ell) = \mathcal{E}(X_n X_{n-\ell}) - 1, \qquad \ell = 0, 1, 2, \cdots, \qquad (4.1)$$

$$\gamma_{YY}(\ell) = \mathcal{E}(Y_n Y_{n-\ell}) - 1, \qquad \ell = 0, 1, 2, \cdots, \qquad (4.2)$$

for the autocovariance functions. Similarly,

$$\gamma_{XY}(\ell) = \mathcal{E}(X_n Y_{n-\ell}) - 1 \qquad \ell = 0, \pm 1, \pm 2, \cdots, \qquad (4.3)$$

is the cross-covariance function.

The following results are immediately obtained using (3.1), (3.2), (4.1) – (4.3) and the stationary assumption:

$$\gamma_{XX}(0) = \gamma_{YY}(0) = 1; \qquad (4.4)$$

$$\gamma_{XY}(0) = \gamma_{YX}(0) = (\alpha_{22}\alpha_{12} + \alpha_{11}\alpha_{21})/\{1 - (\alpha_{22}\alpha_{11} + \alpha_{21}\alpha_{12})\}; \quad (4.5)$$

$$\gamma_{XX}(\ell) = \gamma_{XX}(-\ell) \text{ for all } \ell; \qquad (4.6)$$

$$\gamma_{YY}(\ell) = \gamma_{YY}(-\ell) \text{ for all } \ell; \qquad (4.7)$$

$$\gamma_{XY}(\ell) = \gamma_{YX}(-\ell) \text{ for all } \ell; \qquad (4.8a)$$

$$\gamma_{YX}(\ell) = \gamma_{XY}(-\ell) \text{ for all } \ell. \qquad (4.8b)$$

Autocovariance functons are even functions of the lag ℓ, but the cross-covariance is not.

The range of $\gamma_{XY}(0) = \gamma_{YX}(0)$ as given in (4.5) is the full range of non-negative values up to one. Using the Generalized Reduced Gradient (GRG) algorithm, we find that the maximum takes place along the boundary $\alpha_{11} + \alpha_{12} + \epsilon = 1$ and $\alpha_{22} + \alpha_{21} + \epsilon = 1$ for arbitrary $\epsilon > 0$, in which case we obtained covariances as close to one as desired. Of course, for strictly positive α_{ij} that satisfy the original

constraints, $\gamma_{XY}(0)$ cannot be negative. We show in Section 5 how to incorporate negative correlations into the model.

Using Equations (3.1), (3.2), (4.1)-(4.8) and the stationary assumption, we obtain the following recursion equations for $\gamma_{XX}(\ell)$, $\gamma_{YY}(\ell)$, $\gamma_{XY}(\ell)$, $\gamma_{YX}(\ell)$:

$$\gamma_{XX}(\ell) = \alpha_{11}\gamma_{XX}(\ell-1) + \alpha_{12}\gamma_{YX}(\ell-1), \qquad \ell = 1,2,\cdots, \qquad (4.9)$$

$$\gamma_{YY}(\ell) = \alpha_{22}\gamma_{YY}(\ell-1) + \alpha_{21}\gamma_{XY}(\ell-1), \qquad \ell = 1,2,\cdots, \qquad (4.10)$$

$$\gamma_{YX}(\ell) = \alpha_{21}\gamma_{XX}(\ell-1) + \alpha_{22}\gamma_{YX}(\ell-1), \qquad \ell = 1,2,\cdots, \qquad (4.11)$$

$$\gamma_{XY}(\ell) = \alpha_{12}\gamma_{YY}(\ell-1) + \alpha_{11}\gamma_{XY}(\ell-1), \qquad \ell = 1,2,\cdots. \qquad (4.12)$$

It remains to show that these equations correspond to those given by, for example, Tiao and Box (1981), for the AR(1) Gaussian bivariate time series. We introduce the following matrix notation:

$$\Gamma(\ell) = \begin{bmatrix} \gamma_{XX}(\ell) & \gamma_{XY}(\ell) \\ \gamma_{YX}(\ell) & \gamma_{YY}(\ell) \end{bmatrix}, \qquad \ell = 0,1,2,\cdots, \qquad (4.13)$$

is the covariance matrix for lag ℓ. Equations (3.1) and (3.2) become

$$X_n = K_n X_{n-1} + AE_n, \qquad (4.14)$$

where

$$A = \begin{bmatrix} 1-\alpha_{11}-\alpha_{12} & 0 \\ 0 & 1-\alpha_{22}-\alpha_{21} \end{bmatrix} ; \qquad X_n = \begin{bmatrix} X_n \\ Y_n \end{bmatrix} ;$$

$$E_n = \begin{bmatrix} E_n \\ E_n' \end{bmatrix}$$

and

$$K_n = \begin{bmatrix} K_{11}^{(n)} & K_{12}^{(n)} \\ K_{21}^{(n)} & K_{22}^{(n)} \end{bmatrix} .$$

The matrix-valued random variable K_n assumes 9 different matrix-values independent of X_{n-1} and E_n and retains the property that $\mathcal{E}(K_{ij}^{(n)}) = \alpha_{ij}$ for all i, j, n. Then

$$\mathcal{E}(X_n) = \mathcal{E}(K_n)\mathcal{E}(X_{n-1}) + A\,\mathcal{E}(E_n) \tag{4.15}$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1-\alpha_{11}-\alpha_{12} & 0 \\ 0 & 1-\alpha_{22}-\alpha_{21} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Also,

$$\Gamma(0) = \mathcal{E}(X_n X_n^T) - \mathcal{E}(X_n)\mathcal{E}(X_n^T) = \mathcal{E}(X_n X_n^T) - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \mathcal{E}\{X_n (K_n X_{n-1} + AE_n)^T\} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \mathcal{E}\{X_n (K_n X_{n-1})^T\} + \mathcal{E}\{X_n (AE_n)^T\} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \mathcal{E}\{(K_n X_{n-1} + AE_n)(K_n X_{n-1})^T\} + \mathcal{E}\{(K_n X_{n-1} + AE_n)(AE_n)^T\} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \mathcal{E}\{K_n X_{n-1} (K_n X_{n-1})^T\} + \mathcal{E}\{AE_n (K_n X_{n-1})^T\} + \mathcal{E}\{K_n X_{n-1} (AE_n)^T\}$$

$$+ \mathcal{E}(AE_n E_n^T A^T) - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus

$$\Gamma(0) = \mathcal{E}(K_n X_{n-1} X_{n-1}^T K_n^T) + \mathcal{E}(AE_n X_{n-1} K_n^T) + \mathcal{E}(K_n X_{n-1} E_n^T A^T)$$

$$+ \mathcal{E}(AE_n E_n^T A^T) - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{4.16}$$

Solving (4.16) yields the results given by Equations (4.4) and (4.5). Furthermore

$$\Gamma(1) = \mathcal{E}(X_n X_{n-1}^T) - \mathcal{E}(X_n)\mathcal{E}(X_n^T) = \mathcal{E}(X_n X_{n-1}^T) - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \mathcal{E}\{(K_n X_{n-1} + AE_n)X_{n-1}^T\} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \mathcal{E}(K_n)\Gamma(0) + \mathcal{E}(K_n) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + A\mathcal{E}(E_n X_{n-1}^T) - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus

$$\Gamma(1) = \mathcal{E}(K_n)\Gamma(0) = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \Gamma(0). \tag{4.17}$$

Again

$$\Gamma(-1) = \mathscr{E}(X_n X_{n+1}^T) - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \mathscr{E}\{X_n(K_{n+1}X_n + AE_{n+1})^T\} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \mathscr{E}(X_n X_n^T K_{n+1}^T) + \mathscr{E}(X_n E_{n+1}^T)A^T - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus

$$\Gamma(-1) = \Gamma(0)\mathscr{E}(K_{n+1}^T) = \Gamma(1)^T. \tag{4.18}$$

Clearly $\Gamma(-1)^T = \Gamma(1)$. Because of (4.8a,b) and (4.13), we have

$$\Gamma(-\ell) = \Gamma(\ell)^T \tag{4.19}$$

and

$$\Gamma(\ell) = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}^{\ell} \Gamma(0) \quad \text{for all } \ell = 1, 2, \cdots. \tag{4.20}$$

Now Equation (4.20) is the result given by Tiao and Box, (1981, p. 804). Therefore, the auto- and cross-covariances in this bivariate exponential process in general decay gradually to zero as $|\ell|$ increases, for the values of α_{ij} that satisfy the conditions in Section 3. Numerical examples are given in Section 6, and further discussion of the auto-correlations is presented in Section 7.

In principle, one can use standard estimates of auto- and cross-correlations (Kendall and Stuart, 1983, pp. 661-662) to obtain estimates of $\Gamma(0)$ and $\Gamma(1)$ as

$$C(0) = \begin{bmatrix} 1 & C_{XY}(0) \\ C_{YX}(0) & 1 \end{bmatrix} \qquad (4.21)$$

and

$$C(1) = \begin{bmatrix} C_{XX}(1) & C_{XY}(1) \\ C_{YX}(1) & C_{YY}(1) \end{bmatrix} . \qquad (4.22)$$

Then (4.20) yields estimates of α_{11}, α_{21}, α_{12}, α_{22} as

$$\begin{bmatrix} \hat{\alpha}_{11} & \hat{\alpha}_{12} \\ \hat{\alpha}_{21} & \hat{\alpha}_{22} \end{bmatrix} = \{C(1)\}\{C(0)\}^{-1} \qquad (4.23)$$

Like most moment estimates, these are likely to be fairly poor for small sample size.

An example is given in Table 4.1 where we have performed a simulation of the estimation of α_{11}, α_{12}, α_{21} and α_{22} based on (4.23). Here 10 replications were used and the true values were $\alpha_{11} = 0.70$; $\alpha_{12} = 0.20$; $\alpha_{21} = 0.30$; $\alpha_{22} = 0.50$. The 10 series were run out to $n = 600$ and the first 100 values were discarded as being nonstationary. The boxplots in the tables show the distribution of the tabulated values

in the columns. The estmates show no evidence of departure from normality, although the number of replications is small.

TABLE 4.1

Replication	$\alpha_{11} = 0.70$	$\alpha_{12} = 0.20$	$\alpha_{21} = 0.30$	$\alpha_{22} = 0.50$
1	.63	.32	.27	.59
2	.70	.17	.23	.52
3	.67	.18	.38	.33
4	.71	.09	.29	.32
5	.72	.17	.47	.37
6	.58	.23	.23	.41
7	.74	.22	.41	.38
8	.86	.08	.38	.60
9	.66	.23	.26	.66
10	.70	.20	.22	.50

5. NEGATIVE DEPENDENCE

As pointed out in the last section, the first bivariate exponential model discussed above can produce only positive correlations. Negative correlation is obtainable in one of two ways – through correlation in the bivariate innovation $\{E_n, E_n'\}$ and/or through the correlation in the attenuation controlled by the $K_{ij}^{(n)}$. These ideas are relaxations of the conditions established in Section 3 where $\{E_n\}$ and $\{E_n'\}$ are assumed

to be i.i.d. exponential sequences and $\{K_{11}^{(n)}, K_{12}^{(n)}\}$ is an independent identically distributed bivariate sequence independent of $\{K_{21}^{(n)}, K_{22}^{(n)}\}$ for all n.

It is easy to see that these schemes affect directly only the zero lag covariances, $\Gamma(0)$, and that (4.20) still holds. Thus, it is only necessary here to derive the effect on $\Gamma(0)$ of correlation in the innovation or attenuation sequences.

5.i. Correlated Innovations.

As a first example, consider the model of Section 3 and let

$$
\text{Cov}(E_n, E'_{n+k}) = \begin{cases} \sigma & k=0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}
$$

Now Equation (4.5) becomes

$$
\gamma_{XY}(0) = \frac{(\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22}) + (1-\alpha_{11}-\alpha_{12})(1-\alpha_{21}-\alpha_{22})\sigma}{1 - (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})}. \tag{5.2}
$$

Moran (1967) showed that if $\{E_n\}$ and $\{E'_n\}$ are pairs of exponential random variables with $\lambda = 1$, then $-.6449 \leq \sigma \leq 1$. If $E'_n = a(E_n)$, the antithetic transform of E_n (see for example Gaver and Lewis, 1980), then σ takes on its maximum negative value. It is clear that if $\sigma < 0$, then (5.2) could be negative for some choices of the α_{ij}'s. The process cannot be uncoupled as described previously.

Estimation of σ proceeds as before. The parameters $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ are estimated by (4.23) and then (5.2) can be easily solved for an estimate of σ.

5.11. Correlation in the Random Coefficients.

Consider now the model of Section 3 in the form of (4.14). K_n defines a quadruple where each $K_{ij}^{(n)}$ has a marginal distribution and each row has an independent bivariate distribution for each n. If K_n is now defined so that the rows are not stochastically independent of each other, then (4.5) becomes

$$\gamma_{XY}(0) = \frac{\{(\alpha_{11}\alpha_{21}+\alpha_{12}\alpha_{22}) + 2\sigma_{11,21}+2\sigma_{12,22}+\sigma_{11,22}+\sigma_{12,21}\}}{1 - (\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}) - \sigma_{11,22} - \alpha_{12,21}}. \quad (5.3)$$

where

$$\sigma_{ij,\ell m} = \begin{cases} Cov\{K_{ij}^{(n)}, K_{\ell m}^{(n+k)}\} & k = 0 \\ \\ 0 & \text{otherwise.} \end{cases} \quad (5.4)$$

This correlation must be introduced in such a way that the marginal distributions are not altered. This may not be easy.

The following example using $K_{11}^{(n)} = K_{22}^{(n)}$ for all n demonstrates the above ideas. Define K_n as follows:

$$\begin{bmatrix} K_{11}^{(n)} & K_{12}^{(n)} \\ \\ K_{21}^{(n)} & K_{11}^{(n)} \end{bmatrix} = K_n = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{w.p.} & \alpha_{11}, \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{w.p.} & \alpha_{21}\alpha_{12}, \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \text{w.p.} & \alpha_{12}(1-\alpha_{21}), \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \text{w.p.} & \alpha_{21}(1-\alpha_{12}), \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{w.p.} & 1 - (\alpha_{11}+\alpha_{12}+\alpha_{21}-\alpha_{12}\alpha_{21}). \end{cases} \quad (5.5)$$

where α_{11}, α_{12}, α_{21} as before, and also $0 \leq \alpha_{11}+\alpha_{12}+\alpha_{21}-\alpha_{12}\alpha_{21} \leq 1$. It is easily verified that the marginal distributions of the quadruple are retained and, specifically, that

$$\mathcal{E}(K_n) = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{11} \end{bmatrix} . \tag{5.6}$$

For K_n defined by (5.5) we have, from (5.3)

$$\gamma_{XY}(0) = \frac{\alpha_{11}\{1-(\alpha_{11}+\alpha_{12}+\alpha_{21})\}}{1 - (\alpha_{11}+\alpha_{12}\alpha_{21})} . \tag{5.7}$$

This follows because $\sigma_{11,21} = -\alpha_{11}\alpha_{21}$; $\sigma_{12,22} = -\alpha_{12}\alpha_{11}$; $\sigma_{11,22} = \alpha_{11}(1-\alpha_{11})$ and $\sigma_{12,21} = 0$, with $0 \leq \alpha_{11}+\alpha_{12} \leq 1$; $0 \leq \alpha_{11}+\alpha_{21} \leq 1$; $0 \leq \alpha_{11}+\alpha_{12}+\alpha_{21}-\alpha_{12}\alpha_{21} \leq 1$. Using again GRG, we find $\gamma_{XY}(0)$ attains the full range of positive correlations as before. It has a max negative correlation of $-.125$ at $\alpha_{11} = .25$, $\alpha_{12} = \alpha_{21} = .5$. This corresponds to one of the cases where $K_n \neq 0$ for any n.

Considering the bivariate model with K_n given by (5.5) and with $\alpha_{11} = 0$, we see that $\gamma_{XY}(2k) = 0 = \gamma_{YX}(2k)$, $k = 0,1,2,\cdots$. Resolving (4.9) – (4.12) yields

$$\Gamma(k) = \begin{bmatrix} 0 & \alpha_{12} \\ \alpha_{21} & 0 \end{bmatrix}^k \qquad k = 1,2,\cdots$$

and

$$\Gamma(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

This is the correlation structure of the dual process of Gaver and Lewis (1980). No negative correlation is possible in this configuration.

Lewis and Lawrance (1981) considered the model with correlation in the random coefficients and made $K_{12}^{(n)}$ and $K_{21}^{(n)}$ have the maximum possible negative correlation, obtaining

$$\gamma_{XY}(0) = \frac{\sigma}{1 - (\sigma + \alpha_{12}\alpha_{21})} < 0,$$

where

$$\sigma = \text{Cov}(K_{12}^{(n)}, K_{21}^{(n)}) = \sigma_{12,21} < 0.$$

Considering the bivariate model with K_n given by (5.5) with $\alpha_{12} = \alpha_{21} = 0$, then $\gamma_{XY}(0) = \alpha_{11}$ and the resolution of (4.9) – (4.12) with (5.7) yields the following structure. Marginally $\{X_n\}$ and $\{Y_n\}$ are univariate TEAR(1) processes, since the Y_n's do not appear in the definition of the X_n's and vice versa. Moreover, $\gamma_{XY}(\ell) = \alpha_{11}^{\ell+1}$ $= \gamma_{YX}(\ell)$, $\ell = 0,1,2,\cdots$. Therefore, there is no negative correlation nor any way to uncouple the bivariate process into two independent univariate TEAR(1) processes unless $\alpha_{11} = 0$. Of course, if $\alpha_{11} = 0$, there is then no dependence in the marginal processes either. The bivariate process with $\alpha_{11} \neq 0$ is shown in the next section to have Moran's Bivariate Exponential distribution.

Finally, we observe that if $\alpha_{11} + \alpha_{12} + \alpha_{21} = 1$ in (5.7), we have $\gamma_{XY}(0) = 0$. Thus, the pairs (X_n, Y_n) are uncorrelated for all n and yet the processes $\{X_n\}$ and $\{Y_n\}$ are not at all independent. This can be seen from the fact that $\gamma_{XY}(1) = \alpha_{12} \neq 0$ and $\gamma_{YX}(1) = \alpha_{21} \neq 0$.

5.iii. Correlated Innovations and Random Coefficients.

If there is dependence from both the innovation and cross-coupled attenuation, then a general expression for (4.5) becomes

$$\gamma_{XY}(0)=\frac{\{(\alpha_{21}\alpha_{11}+\alpha_{22}\alpha_{12})+(2\sigma_{11,21}+2\sigma_{12,22}+\sigma_{11,22}+\sigma_{12,21})+(1-\alpha_{11}-\alpha_{12})(1-\alpha_{21}-\alpha_{22})\sigma\}}{1-(\alpha_{12}\alpha_{21}+\alpha_{22}\alpha_{11})-(\sigma_{12,21}+\sigma_{11,22})}$$

(5.8)

The interesting point to be made here is that the effects are additive in the numerator of (5.8). This general model is called the BEAR(1) model.

6. NUMERICAL EXAMPLES OF COVARIANCE MATRICES.

The form of $\Gamma(\ell)$ given in (4.20) can be misleading. Since the form of $\Gamma(\ell)$ is analogous to that of a univariate AR(1) process, one might think that the elements of $\Gamma(\ell)$ should experience a geometric decay as $|\ell|$ increases. In fact, auto- and cross-covariances can exhibit many different patterns depending on the innovation and the attenuation structures of the model. Some illustrative examples follow.

In Table 6.1 an example is described where $\gamma_{XY}(0)$ is given by (4.5). Although the auto-covariance functions decrease monotonically, it is not at all like a geometric decay.

In Table 6.2, an example corresponding to $\gamma_{XY}(0)$ as given in (5.2) is exhibited. This case demonstrates that negative correlation in the bivariate innovation can create negative cross-covariances at lag zero. It can also cause oscillation in the values of the auto- and cross-covariance functions at subsequent lags. Also, note for this example

that simply by interchanging α_{11} for α_{12} and α_{21} for α_{22}, we can create another bivariate series with the same $\Gamma(0)$, but with auto-covariance functions $\gamma_{XX}(\ell)$ and $\gamma_{YY}(\ell)$ that are almost geometrically decaying.

In Table 6.3, we give a final example corresponding to $\gamma_{XY}(0)$ from (5.3). It is, in fact, the example given for K_n in (5.5). Again, $\gamma_{XY}(0)$ is negative. However, note that although the auto-covariance functions are generally decreasing, the movement is not monotone as in Table 6.1, and certainly not decreasing geometrically.

It is apparent from all examples that as $|\ell|$ increases, $\Gamma(\ell)$ approaches the zero matrix. In fact, since the eigenvalues of the matrix are less than one, the components of $\Gamma(\ell)$ all decrease to zero.

TABLE 6.1 Covariance Matrix Corresponding to (4.5)

	$\alpha_{11} = .2$	$\alpha_{12} = .7$	$\alpha_{21} = .3$	$\alpha_{22} = .5$
ℓ	$\gamma_{XX}(\ell)$	$\gamma_{YX}(\ell)$	$\gamma_{XY}(\ell)$	$\gamma_{YY}(\ell)$
0	1.0000	0.5942	0.5942	1.0000
1	0.6159	0.5971	0.8188	0.6783
2	0.5412	0.4833	0.6386	0.5848
3	0.4466	0.4040	0.5371	0.4840
4	0.3721	0.3360	0.4462	0.4031
5	0.3096	0.2796	0.3714	0.3354
6	0.2577	0.2327	0.3091	0.2791
7	0.2144	0.1936	0.2572	0.2323
8	0.1784	0.1611	0.2140	0.1933
9	0.1485	0.1341	0.1781	0.1609
10	0.1236	0.1116	0.1482	0.1339
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
25	0.0089	0.0081	0.0106	0.0097

TABLE 6.2 Covariance Matrix Corresponding to (5.2)

$$\alpha_{11} = .05 \qquad \alpha_{12} = .60 \qquad \alpha_{21} = .40 \qquad \alpha_{22} = .01$$
$$\sigma = -.6449$$

ℓ	$\gamma_{XX}(\ell)$	$\gamma_{YX}(\ell)$	$\gamma_{XY}(\ell)$	$\gamma_{YY}(\ell)$
0	1.0000	−0.1411	−0.1411	1.0000
1	−0.0347	0.3986	0.5929	−0.0464
2	0.2374	−0.0099	0.0018	0.2367
3	0.0059	0.0949	0.1421	0.0031
4	0.0572	0.0033	0.0090	0.0569
5	0.0048	0.0229	0.0346	0.0042
6	0.0140	0.0021	0.0043	0.0139
7	0.0020	0.0056	0.0086	0.0019
8	0.0035	0.0009	0.0016	0.0035
9	0.0007	0.0014	0.0022	0.0007
10	0.0009	0.0002	0.0005	0.0009

TABLE 6.3 Covariance Matrix Corresponding to (5.3)

$$\alpha_{11} = .25 \qquad \alpha_{12} = .60 \qquad \alpha_{21} = .40 \qquad \alpha_{22} = .25$$
$$\sigma_{11,22} = -.1 \quad \sigma_{12,22} = -.15 \quad \sigma_{11,22} = .1875 \quad \sigma_{12,21} = 0$$

ℓ	$\gamma_{XX}(\ell)$	$\gamma_{YX}(\ell)$	$\gamma_{XY}(\ell)$	$\gamma_{YY}(\ell)$
0	1.0000	−0.1225	−0.1225	1.0000
1	0.1765	0.3694	0.5694	0.2010
2	0.2659	0.1629	0.2629	0.2780
3	0.1642	0.1470	0.2325	0.1747
4	0.1293	0.1024	0.1629	0.1367
5	0.0938	0.0773	0.1227	0.0993
6	0.0698	0.0568	0.0903	0.0739
7	0.0516	0.0421	0.0669	0.0546
8	0.0382	0.0312	0.0495	0.0404
9	0.0282	0.0231	0.0366	0.0299
10	0.0209	0.0171	0.0271	0.0221

7. THE MARGINAL PROCESS.

Writing $K = \mathcal{E}(K_n) = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$, we have from (4.20) the relationship

$$\Gamma(\ell) = K^{\ell}\Gamma(0). \tag{7.1}$$

Further, since K is a square matrix, the well-known Caylay-Hamilton Theorem proivdes that there exists constants ϕ_1 and ϕ_2, such that

$$K^2 - \phi_1 K - \phi_2 I = 0.$$

Applying this result to (7.1) yields

$$\Gamma(\ell) - \phi_1\Gamma(\ell-1) - \phi_2\Gamma(\ell-2) = 0, \quad \ell \geq 2. \tag{7.2}$$

The difference Equation (7.2) applies to each element of the matrix $\Gamma(\ell)$. Hence, for example, the autocorrelation function of $\{X_n\}$ satisfies

$$\rho_{XX}(\ell) - \phi_1\rho_{XX}(\ell-1) - \phi_2\rho_{XX}(\ell-2) = 0, \quad \ell \geq 2.$$

Thus, in general, $\{X_n\}$ has the correlation structure of an ARMA(2,1) process. In addition, the lag 1 autocorrelation may be derived from

(7.1) when $\ell = 1$. In particular, $\gamma_{XX}(1) = \alpha_{11}\gamma_{XX}(0) + \alpha_{12}\gamma_{YX}(0)$, and since $\gamma_{XX}(0) = \gamma_{YY}(0) = 1$, we find that $\rho_{XX}(1) = \alpha_{11} + \alpha_{12}\rho_{YX}(0)$.

Further, the constants ϕ_1 and ϕ_2 may be obtained in terms of the elements of K. It may be shown directly that $\phi_1 = (\alpha_{11}+\alpha_{22})$ and $\phi_2 = (\alpha_{12}\alpha_{21}-\alpha_{11}\alpha_{22})$. Thus, in summary, the autocorrelation functions for the marginal processes $\{X_n\}$, $\{Y_n\}$ satisfy

$$\rho(\ell) = (\alpha_{11}+\alpha_{22})\rho(\ell-1) + (\alpha_{12}\alpha_{21}-\alpha_{11}\alpha_{22})\rho(\ell-2), \qquad (7.3)$$

for $\ell \geq 2$, and

$$\rho_{XX}(1) = \alpha_{11} + \alpha_{12}\rho_{YX}(0), \qquad (7.4)$$

$$\rho_{YY}(1) = \alpha_{22} + \alpha_{21}\rho_{XY}(0). \qquad (7.5)$$

In general, $\{X_n\}$ and $\{Y_n\}$ have correlation structures of ARMA(2,1) processes. In any particular case, their actual form will depend upon the values $\{\alpha_{ij}\}$. The marginal processes will have ARMA(p,q) correlations, where $p \leq 2$ and $q \leq 1$. We illustrate these ideas with some examples.

(i) Using $\alpha_{12} = \alpha_{21} = 0$ results in the X and Y processes being independent TEAR(1) processes. From (7.4) and (7.5), $\rho_{XX}(1) = \alpha_{11}$, $\rho_{YY}(1) = \alpha_{22}$, and using (7.3) we obtain $\rho_{XX}(\ell) = \alpha_{11}^{\ell}$ and $\rho_{YY}(\ell) = \alpha_{22}^{\ell}$. The bivariate system has become

$$X_n = K_{11}^{(n)}X_{n-1} + (1-\alpha_{11})E_n,$$

$$Y_n = K_{22}^{(n)} Y_{n-1} + (1-\alpha_{22}) E_n',$$

where $P(K_{11}^{(n)}=1) = \alpha_{11}$ and $P(K_{22}^{(n)}=1) = \alpha_{22}$, and $K_{11}^{(n)}$ and $K_{22}^{(n)}$ are independent, as are E_n and E_n'.

(ii) Using $\alpha_{11} = \alpha_{22} = 0$ leads to coupled processes, each with AR(1) correlation structure. Now, $\rho_{XX}(1) = \alpha_{12}\rho$ and $\rho_{YY}(1) = \alpha_{21}\rho$, where $\rho = \rho_{YX}(0)$. From (7.3) we may derive

$$\left. \begin{array}{l} \rho_{XX}(2k) = (\alpha_{12}\alpha_{21})^k \\[2ex] \rho_{XX}(2k+1) = (\alpha_{12}\alpha_{21})^k \alpha_{12}\rho \end{array} \right] \quad k = 0,1,\cdots.$$

This is the formulation used by Gaver and Lewis (1980) to derive negatively correlated processes. In effect, they chose $\alpha_{12} = \alpha_{21} = \alpha$, and so

$$\rho_X(k) = \left[\begin{array}{ll} \alpha^k & k \text{ even} \\[2ex] \alpha^k \rho & k \text{ odd.} \end{array} \right.$$

Now, from (5.8) or (5.2) with the choice of K used here, $\rho = \rho_{YX}(0) = (1-\alpha)\rho_{EE'}(0)/(1+\alpha)$, and choosing E_n and E_n' to be negatively correlated results in a sign-switching autocorrelation for $\{X_n\}$ and $\{Y_n\}$.

(iii) Using $\alpha_{11} = \alpha_{21} = 0$ may yield a process with ARMA(1,1) correlation structure. The bivariate system is now

$$X_n = K_{12}^{(n)} Y_{n-1} + (1-\alpha_{12})E_n$$

$$Y_n = K_{22}^{(n)} Y_{n-1} + (1-\alpha_{22})E_n'. \tag{7.6}$$

From (7.4) and (7.5) we have $\rho_{XX}(1) = \alpha_{12}\rho$ and $\rho_{YY}(1) = \alpha_{22}$, where $\rho = \rho_{YX}(0)$, as before. Further, Equation (7.3) reduces to $\rho(\ell) = \alpha_{22}\rho(\ell-1)$, $\ell \geq 2$. Hence, $\rho_{YY}(\ell) = \alpha_{22}^{\ell}$, $\ell \geq 0$, and $\rho_{XX}(\ell) = (\alpha_{12}\rho)\alpha_{22}^{\ell-1}$, $\ell \geq 1$. Thus, $\{Y_n\}$ is a process with AR(1) correlation structure, as may be seen directly from (7.6). In addition, $\{X_n\}$ is a process with ARMA(1,1) correlation structure, provided $\rho_{YX}(0) \neq 0$. A particularly simple way to achieve this is to take $E_n' = E_n$. The process given by (7.6) is then very closely related to the EARMA(1,1) and DARMA(1,1) processes of Jacobs and Lewis (1977,1983). On the other hand, we can choose E_n' negatively correlated with E_n, e.g., its antithetic, and may then induce negative correlation in $\{X_n\}$.

(iv) In order to generate the autocorrelation function corresponding to an AR(2) process, it is necessary that $\rho(\ell)$ satisfy the difference Equation (7.3) for $\ell = 1$ also. Using $\rho(-1) = \rho(1)$ this condition becomes $\rho(1) = \phi_1/(1-\phi_2)$. Rewriting this with the appropriate values for ϕ_1, ϕ_2, and using the definition of $\rho_{XX}(1)$ given by (7.4) yields the condition

$$\rho = \frac{\alpha_{22} + \alpha_{11}(\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})}{\alpha_{12}(1+\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})}, \tag{7.7}$$

where $\rho = \rho_{YX}(0)$, as before. In a model in which X_n and Y_n are independent the condition becomes

$$\alpha_{22} + \alpha_{11}(\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}) = 0. \qquad (7.8)$$

In this latter case, the condition (7.8) may be satisfied by choosing either: (a) $\alpha_{11} = 1$; or (b) $\alpha_{22} = 0$ and $\alpha_{11}\alpha_{12}\alpha_{21} = 0$. Case (a) leads to example (i) above, i.e., two independent processes, each with AR(1) correlation structure, one of which is degenerative, $X_n = X_{n-1}$. Case (b) leads to a variety of possible models similar to those discussed in the previous examples, and one we have not noted yet, a process with Moving Average of order 1, MA(1), correlation structure. If $\alpha_{11} = \alpha_{21} = \alpha_{22} = 0$, the bivariate process is

$$X_n = K_{12}^{(n)} Y_{n-1} + (1 - \alpha_{12})E_n$$
$$Y_n = E_n'.$$

Thus, if $E_n = E_n'$, then $\{X_n\}$ has the correlation structure of a MA(1) process and $\{Y_n\}$ is a sequence of i.i.d. exponentials.

Hence, we cannot derive an AR(2) structure using (7.8). We require dependence between X_n and Y_n and the satisfaction of condition (7.7). Since the relationship between $\rho_{YX}(0)$ and $\rho_{EE'}(0)$ given by (5.2) and the condition (7.7) are fairly complex, we shall not attempt any general analysis. We note only that it is certainly possible and give an example.

If (E_n, E_n') have an instantaneous correlation of 0.5, then $\rho = 4/17$, from (5.2), and (7.7) is satisfied by $\alpha_{11} = 0$, $\alpha_{12} = 0.5$, $\alpha_{21} = 0.3$ and $\alpha_{22} = 0.1$. This choice of K and the correlation between E_n and E_n' yields a process $\{X_n\}$ whose autocorrelation function satisfies

$$\rho_{XX}(\ell) = 0.1\rho_{XX}(\ell-1) + 0.15\rho_{XX}(\ell-2),$$

for all $\ell \geq 1$. In passing, we may note that $\{Y_n\}$ in this process has the correlation structure of an ARMA(2,1) process.

We now consider the general case, i.e., the process with ARMA(2,1) correlation structure, and note some of its properties here. Solutions of the difference Equation (7.2) depend upon the behavior of the roots of the quadratic equation $Z^2 - \phi_1 Z - \phi_2 = 0$. It is easily verified that with our restrictions on $\{\alpha_{ij}\}$ we have: $\phi_1 + \phi_2 < 1$, $0 < \phi_1 < 2$ and $-1 < \phi_2 < 1$. These ensure that the roots of the quadratic lie within the unit circle (See Box and Jenkins, 1976, p. 58–59, for a discussion of this). However, we may also demonstrate that the roots are real. The condition for this is $\phi_1^2 + 4\phi_2 \geq 0$, and

$$\phi_1^2 + 4\phi_2 = (\alpha_{11} + \alpha_{22})^2 + 4(\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})$$

$$= (\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}\alpha_{21} \geq 0.$$

Figure 7.1 shows the region in which (ϕ_1, ϕ_2) will lie for these processes. It is worth comparing this region with the more general one available for a stationary process with an AR(2) component. It is given

in Box and Jenkins (1976, p. 59), and is defined by $\phi_1 + \phi_2 < 1$, $\phi_1 - \phi_2 < -1$, $-1 < \phi_2 < 1$. In our case, only positive ϕ_1 is possible, and the characteristic quadratic has only real roots. The most important effect of this latter property is that the autocorrelations are all positive if $\rho(1)$ is positive. We have seen in the examples above that this restriction is not necessary in the cases when the autoregressive component is first order. Also, we can introduce negative autocorrelation at lag one using (7.4) and (7.5) and negatively correlated (E_n, E_n'). This may persist for higher lags depending on ϕ_1 and ϕ_2 as in the numerical examples.
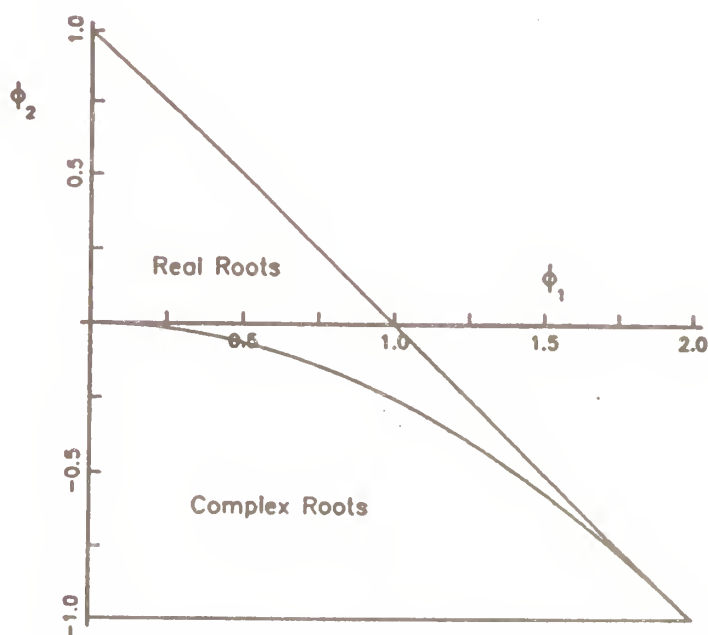

Figure 7.1. Region of (ϕ_1, ϕ_2) for the bivariate exponential process.

8. JOINT DISTRIBUTION OF (X_n, Y_n)

We turn our attention to the formulation of the joint distribution of (X_n, Y_n) by investigating the joint Laplace-Stieltjes transform. For the initial bivariate model described in (3.1) and (3.2), the expression for the transform

$$\varphi_{X_n Y_n}(s, t) = \mathscr{E}\{\exp(-sX_n - tY_n)\}, \tag{8.1}$$

is very complicated.

However, two special cases are easily verified. If $K_{12}^{(n)} = K_{21}^{(n)} = 0$ for all n in (3.1) and (3.2), we have after considerable simplification

$$\varphi_{XY}(s, t) = \left[\frac{1}{1+s}\right]\left[\frac{1}{1+t}\right] . \tag{8.2}$$

This tallies with the result of Section 3 that under these conditions $\{X_n\}$ and $\{Y_n\}$ are independent univariate TEAR(1) processes, the first with correlation structure $\rho_{11}(k) = \alpha_{11}^k$, the second with $\rho_{22}(\ell) = \alpha_{22}^\ell$. If, in addition, $K_{11}^{(n)} = K_{22}^{(n)} = K_n$ and

$$K_n = \begin{bmatrix} 1 & \text{w.p.} & \alpha \\ 0 & \text{w.p.} & 1-\alpha \end{bmatrix}. \tag{8.3}$$

then the distribution of (X_n, Y_n) turns out to be Moran's Bivariate Exponential distribution. Using (3.1) and (3.2), we have from (6.1)

$$\varphi_{X_n Y_n}(s, t) = \mathscr{E}\left[\exp[-s\{K_n X_{n-1} + (1-\alpha)E_n\} - t\{K_n Y_{n-1} + (1-\alpha)E_n'\}]\right] . \tag{8.4}$$

Using the independence of $\{E_n\}$ and $\{E'_n\}$, the stationarity of $\{X_n, Y_n\}$ and conditioning on K_n, we have after simplification

$$\varphi_{XY}(s,t) = \frac{1}{(1+s)(1+t) - \alpha st} \cdot \qquad (8.5)$$

Equation (8.5) is the Laplace-Stieltjes transform for the Moran Bivariate Exponential Distribution as given in Johnson and Kotz (1970, p. 267).

9. CONCLUSIONS

The NEAR(2) model has been demonstrated to yield a Bivariate First Order Autoregressive process with exponential marginals by cross-coupling and auto-coupling the two marginal processes. The process has the same correlation structure as the Gaussian Bivariate AR(1) process. The results for another bivariate process with exponential marginals that was proposed by Raftery (1982) using the NEAR(1) structure of Lawrance and Lewis (1981) hold only in very special cases. Even then, as we have shown, the correlation structure is identical to the Gaussian AR(1) model.

The possibility of negative correlations was explored using the ideas of correlated innovation and/or cross-correlated attenuation. These situations occur frequently in modelling physical phenomena. For example, the same shock at time n to a system produces related effects in components given by X_n and Y_n. Likewise, if X_n and Y_n are flows in a river at two different points along a bank, then $E_n = E'_n$ represents a common phenomenon driving both series. Finally, when both

series are TEAR(1) series linked by a common attenuation and independent innovations, the bivariate distribution of (X_n, Y_n) was shown to be the familiar Moran distribution.

Taken together these options include numerous possibilities to model exponential bivariate time series. There is still much work in parameter estimation before these models can be widely applied. They certainly lend themselves at this time to an analysis via simulation.

One other detail which could extend the utility of the model needs to be pointed out. This is that Y_{n-1} in (3.1) could be replaced by $Y_{n-\ell}$, and X_{n-1} in (3.2) could be replaced by $X_{n-\ell'}$. The model is still well defined and has obvious physical interpretation. Properties of this extension of the model will be addressed elsewhere.

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11. ACKNOWLEDGEMENTS

APL Graphics program GRAFSTAT from IBM Research. This **program is** available at the Naval Postgraduate School under a test-site **agreement** with IBM Research.

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